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COEFFICIENT INVERSE HEAT-CONDUCTION PROBLEM

E. A. Artyukhin and A. V. Nenarokomov

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The computational algorithm and the results are given for the solution of the inverse problem of determining the total set of coefficients of the inhomogeneous quasilinear heat-conduction equation.

Recently, nonsteady experimental—computational methods based on the solution of the coefficient (in the terminology of [1]) inverse heat-conduction problems (IHP) have been sufficiently widely used to determine the thermophysical characteristics of various structural and heat-protective materials. Expansion of the range of practical application of such methods is directly associated with the development of effective computational algorithms for the solution of nonlinear multiparameter inverse problems in which a whole set of unknown characteristics is determined from the data of a single nonsteady experiment. This type of algorithm may ensure maximum information retrieval from thermophysical experiments.

Consider a one-dimensional heat-transfer process with a mathematical model in the form of a boundary problem for the quasilinear inhomogeneous heat-conduction equation

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)+S(T),\ x,\ \tau\in Q=(0,\ b)\times(0,\ \tau], \tag{1}$$

$$T(x,\ 0)=T_0(x),\ x\in[0,\ b], \tag{2}$$

$$\gamma_1\lambda(T(0,\ \tau))\frac{\partial T(0,\ \tau)}{\partial x}+\mu_1 T(0,\ \tau)=g_1(\tau),\ \tau\in(0,\ \tau_m], \tag{3}$$

$$\gamma_2\lambda(T(b,\ \tau))\frac{\partial T(b,\ \tau)}{\partial x}+\mu_2 T(b,\ \tau)=g_2(\tau),\ \tau\in(0,\ \tau_m], \tag{4}$$

where $T_0(x)$, $g_1(\tau)$, $g_2(\tau)$ are known functions; b, τ_m, μ_1, μ_2, γ_1, γ_2 are specified numbers.

Suppose that thermosensors are placed at some number (N + 2) of points in the interval [0, b] with coordinates $x = X_i$, $i = \overline{1, N}$, $0 = X_0 < X_1 < \ldots < X_N < X_{N+1} = b$, and dynamic temperature measurements are undertaken

$$T^{\exp}(X_i,\ \tau)=f_i(\tau),\ i=\overline{0,\ N+1}. \tag{5}$$

It is assumed here that, if a boundary condition of the first kind is imposed at any boundary, the functions $g_j(\tau)$, j = 1, 2 in Eqs. (3) and (4) are formed on the basis of the data of the corresponding measurements $g_1(\tau) = f_0(\tau)$, $g_2(\tau) = f_{N+1}(\tau)$. Depending on a priori information on the characteristics C(T), λ(T), and S(T), different formulations of the coefficient IHP are possible: the derivation of any one characteristic or some set of characteristics simultane-

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ously. From the viewpoint of maximum information content of the thermophysical investigation, it is of interest to solve the inverse problem of determining the whole set C(T), λ(T), and S(T). This problem is considered in the present work.

The inverse problem which is analyzed may be written in the form of an operator equation of the first kind

$$Au = f,\ u \in U,\ f \in F,\ A: U \to F, \tag{6}$$

where $u = \{u_l(T)\}_1^3 = \{C(T), \lambda(T), S(T)\}$ is the desired vector function; A is a nonlinear operator subject to the boundary problem in Eqs. (1)-(4) and corresponding to each element $u \in U$ of the spur of the solution in Eqs. (1)-(4) $T(X_i, \tau)$, $i = \overline{0, N+1}$. In the numerical solution of Eq. (6), the operator A is usually constructed by finite-dimensional approximation of the problem in Eqs. (1)-(4) and hence is known with an error. The right-hand side f is formed using experimental data and is therefore also specified approximately. In this case, the problem in Eq. (6) is incorrect [2]. Its accurate solution is the element $u \in U$ for which the spur of the solution of the boundary condition in Eqs. (1)-(4) coincides with the specified right-hand side of Eq. (6), $f \in F$. The space L_2 is usually taken as F. With inconsistent specification of the initial data, there may be no accurate solution and, if it does exist, it will not have the property of stability relative to errors in the initial data [2].

In view of the incorrectness of the given inverse problem, it must be solved using special regularizing algorithms [2]. One effective and universal method of constructing such algorithms is iterative regularization [1]. In this case, any first-order gradient method, for example, the method of fastest descent or conjugate gradients, is used to construct the sequence minimizing the discrepancy functional $J(u) = \|Au - f\|_F$

$$u^{r+1} = u^r + \alpha^r G(J'^{(r)}),\ r = 0, 1, \ldots, R, \tag{7}$$

where u^0 is the initial approximation; α^r is the depth of descent, chosen from the condition $J^{r+1} = \min_{\alpha \in R^+} J(u^r + \alpha G(J'^{(r)}))$; G(J') is the direction of descent; R is the number of the last iteration, determined in the course of solution from the discrepancy condition $J(u^R) \approx \delta$; δ is the known integral error of the initial data.

For linear incorrect problems of the type in Eq. (6), in the presence of error in the operator A and the right-hand side f, the present method was given a rigorous mathematical basis in [3, 4]. For the solution of coefficient IHP, which are always nonlinear, the high efficiency of the iterative algorithms was demonstrated in [5-8], for example, where the derivation of temperature dependences of one or two coefficients of the homogeneous heat-conduction equation was considered.

Writing the mean square discrepancy

$$J = \sum_{i=0}^{N+1} \int_0^{\tau_m} [T(X_i, \tau) - f_i(\tau)]^2 d\tau \tag{8}$$

the iterative algorithm for solution of the above-formulated inverse problem is constructed. Conditions ensuring unique solution of the problem are satisfied here [9, 10].

One of the central problems characterizing the computational efficiency of the iterative algorithms is to determine the gradient of the functional in Eq. (8). In solving coefficient inverse problems in which the desired characteristics depend on the solution of the direct problem in Eqs. (1)-(4), it is impossible to construct an effective procedure for minimizing the discrepancy in Eq. (8) in functional space. At the same time, the approach based on parameterization of unknown functions is very productive. As a result, the inverse problem is reduced to seeking the vector of unknown parameters including the coefficients of parametric representation of all the desired functions.

The components of the unknown function $u = \{u_l(T)\}_1^3$ are written in the form

$$u_l(T) = \sum_{k=1}^{m_l} p_k \varphi_{l_k}(T),\ l = \overline{1, 3}, \tag{9}$$

where $\varphi_{lk}(T)$, $k = \overline{1, m_l}$, $l = \overline{1, 3}$ are specified systems of basis functions. Then, following the approach of [6], it may be shown that the gradient of the functional in Eq. (8) is determined by the following formulas

$$J'_{u_{1k}} = -\int_0^{\tau_m}\int_0^b \psi \frac{\partial T}{\partial \tau} \varphi_{1k}(T)\,dx d\tau,\ k = \overline{1, m_1}; \tag{10}$$

$$J'_{u_{2k}} = \int_0^{\tau_m}\int_0^b \psi\left[\frac{\partial^2 T}{\partial x^2}\varphi_{2k}(T) + \left(\frac{\partial T}{\partial x}\right)^2 \frac{d\varphi_{2k}}{dT}\right] dx d\tau + \int_0^{\tau_m} \psi(0, \tau)\frac{\partial T(0, \tau)}{\partial x}\varphi_{2k}(T(0, \tau))\,d\tau -$$

$$-\int_0^{\tau_m} \psi(b, \tau)\frac{\partial T(b, \tau)}{\partial x}\varphi_{2k}(T(b, \tau))\,d\tau,\ k = \overline{1, m_2}; \tag{11}$$

$$J'_{u_{3k}} = \int_0^{\tau_m}\int_0^b \psi\varphi_{3k}(T)\,dx d\tau,\ k = \overline{1, m_3}, \tag{12}$$

where $\psi(x, \tau)$ is the solution of the boundary problem for the conjugate variable

$$-C(T)\frac{\partial \psi_i}{\partial \tau} = \lambda(T)\frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial S}{\partial T},$$

$$x, \tau \in Q_i - (X_{i-1}, X_i) \times (0, \tau_m],\ i = \overline{1, N+1}; \tag{13}$$

$$\psi_i(x, \tau_m] = 0,\ x \in [X_{i-1}, X_i],\ i = \overline{1, N+1}; \tag{14}$$

$$\gamma_1\lambda(T(0, \tau))\frac{\partial \psi_1(0,\tau)}{\partial x} + \mu_1\psi_1(0, \tau) = 2[T(0, \tau) - f_0(\tau)],\ \tau \in [0, \tau_m); \tag{15}$$

$$\psi_i(X_i, \tau) = \psi_{i+1}(X_i, \tau),\ i = \overline{1, N},\ \tau \in [0, \tau_m), \tag{16}$$

$$\lambda(T(X_i, \tau))\left[\frac{\partial \psi_i(X_i, \tau)}{\partial x} - \frac{\partial \psi_{i+1}(X_i, \tau)}{\partial x}\right] = 2[T(X_i, \tau) - f_i(\tau)],$$

$$i = \overline{1, N},\ \tau \in [0, \tau_m); \tag{17}$$

$$\gamma_2\lambda(T(b, \tau))\frac{\partial \psi_{N+1}(b, \tau)}{\partial x} + \mu_2\psi_{N+1}(b, \tau) = 2[T(b, \tau) - f_{N+1}(\tau)],\ \tau \in [0, \tau_m). \tag{18}$$

In determining the depth of search α^r in iterative algorithms for solving multiparameter inverse problems, it is very effective to represent this quantity in the form of a vector of dimensionality corresponding to that of the desired solution of the inverse problem and to calculate the linear estimate for each component [11]. Using this approach, α is written in the form

$$\alpha = \{\alpha_1, \alpha_2, \alpha_3\}. \tag{19}$$

The components of the vector in Eq. (19) are calculated from the solution of the following system of linear algebraic equations

$$\sum_{j=1}^{3}\alpha_j\left(\sum_{i=0}^{N+1}\int_0^{\tau_m}\vartheta_j(X_i, \tau)\vartheta_k(X_i, \tau)\,d\tau\right) = -\sum_{i=0}^{N+1}\int_0^{\tau_m}[T(X_i, \tau) - f_i(\tau)]\,\vartheta_k(X_i, \tau)\,d\tau,\ k = \overline{1, 3}, \tag{20}$$

where the functions $\vartheta_j(x, \tau)$ satisfy the boundary problem

$$C(T)\frac{\partial \vartheta_j}{\partial \tau} = \lambda(T)\frac{\partial^2 \vartheta_j}{\partial x^2} + 2\frac{\partial \lambda}{\partial T}\frac{\partial T}{\partial x}\frac{\partial \vartheta_j}{\partial x} + \left[\frac{\partial^2 T}{\partial x^2}\frac{\partial \lambda}{\partial T} +\right.$$

$$\left. + \left(\frac{\partial T}{\partial x}\right)^2\frac{\partial^2 \lambda}{\partial T^2} + \frac{\partial S}{\partial T} - C(T)\frac{\partial T}{\partial \tau}\right]\vartheta_j + W_j,\ x, \tau \in Q; \tag{21}$$

$$\vartheta_j(x, 0) = 0,\ x \in [0, b]; \tag{22}$$

$$\gamma_1\left[\lambda(T(0, \tau))\frac{\partial \vartheta_j(0, \tau)}{\partial x} - \frac{\partial T(0, \tau)}{\partial x}\frac{\partial \lambda(T(0, \tau))}{\partial T}\vartheta_j(0, \tau)\right] +$$

$$+ p_j + \mu_1\vartheta_j(0, \tau) = 0,\ \tau \in (0, \tau_m]; \tag{23}$$

$$\gamma_2\left[\lambda(T(b,\tau))\frac{\partial\vartheta_j(b,\tau)}{\partial x}+\frac{\partial T(b,\tau)}{\partial x}\frac{\partial\lambda(T(b,\tau))}{\partial T}\vartheta_j(b,\tau)\right]+p_j+\mu_2\vartheta_j(b,\tau)=0,\ \tau\in(0,\tau_m], \tag{24}$$

where

$$W_j=\begin{cases}-\dfrac{\partial T}{\partial\tau}\sum\limits_{k=1}^{m_1}J'_{u_{1_k}}\varphi_{1_k}(T)\ \text{when}\ j=1,\\ \dfrac{\partial^2T}{\partial x^2}\sum\limits_{k=1}^{m_2}J'_{u_{2_k}}\varphi_{2_k}(T)+\left(\dfrac{\partial T}{\partial x}\right)^2\sum\limits_{k=1}^{m_2}J'_{u_{2_k}}\dfrac{d\varphi_{2_k}}{\partial T}\ \text{when}\ j=2,\\ \sum\limits_{k=1}^{m_3}J'_{u_{3_k}}\varphi_{3_k}(T)\ \text{when}\ j=3,\end{cases}$$

$$p_j=\begin{cases}\dfrac{\partial T}{\partial x}\sum\limits_{k=1}^{m_2}J'_{u_{2_k}}\varphi_{2_k}(T)\ \text{when}\ j=2,\\ 0\qquad\text{when}\ j=1,\ 3.\end{cases}$$

The system in Eq. (20) has a symmetric matrix and may be solved by one of the well-known methods, for example, the square-root method [12].

Note that the region of definition of the unknown functions is not known in advance, and must be refined at each iteration.

A computational algorithm is now developed in accordance with the given scheme of solution of the coefficient IHP. The boundary problems in Eqs. (1)-(4), (13)-(18), and (21)-(24) are replaced by the corresponding finite-difference approximations on space—time grids. An implicit approximation scheme is used, with iteration with respect to the coefficients [13], until the solutions coincide with the relative accuracy ε_0 specified a priori.

The algorithm for deriving the volume specific heat, thermal conductivity, and heat-source coefficient is realized in the form of a computer program, which is used to calculate a series of methodological examples.

The solution of the IHP is modeled in the following traditional manner. The dependence of the coefficients of the heat-conduction equation on the temperature is specified, along with the initial and boundary conditions, and the direct problem is solved. Using the calculated temperature field, the readings of thermosensors "positioned" at several points of the spatial grid are formed. Then the inverse problem is solved under the assumption that the coefficients of the heat-conduction equation are unknown.

With the aim of matching the errors of the finite-different approximation, the boundary problems in Eqs. (1)-(4), (13)-(18), and (21)-(24) are solved on the same space—time grid. The number of steps is chosen parametrically: the direct problem in Eqs. (1)-(4) is solved successively with increase in the number of steps, until the solutions coincide with accuracy ε_0 equal to the accuracy with which the solutions coincide in iteration with respect to the coefficients. Some of the results obtained are shown in Fig. 1.

Consider the nonsteady heating of an infinite plate of thickness b = 0.03 m for τ_m = 21 sec. The initial temperature distribution is taken to be constant at T_0 = 300°K. One boundary is heat-insulated: $\mu_2 = 0$, $\gamma_2 = -1$, $g_2 \equiv 0$; the other is subjected to intense thermal perturbations ($\mu_1 = 0$, $\gamma_1 = -1$)

$$g_1(\tau)=0{,}019\cdot10^7\tau,\ \text{W/m}^2. \tag{25}$$

The specified coefficient values for the heat-conduction equation are given in Fig. 1. The relative accuracy of the output from iterations with respect to the coefficients is 0.001.

As the basis functions in Eq. (9), cubic B splines are used [14]. The unknown functions are approximated on the segment [T_{min}, T_{max}] = [300°K, 1200°K], using three division sections. The choice of the number of sections is made parametrically.

The IHP is solved with "measurements" of the temperature by three thermosensors with coordinates X_0 = 0, X_1 = 0.022, X_2 = 0.03 m on the difference grid $n_\tau \times n_x$ = 50 × 42. The iterative process ends when the derived characteristics in adjacent iterations coincide with a specified relative accuracy ε_f = 0.01.

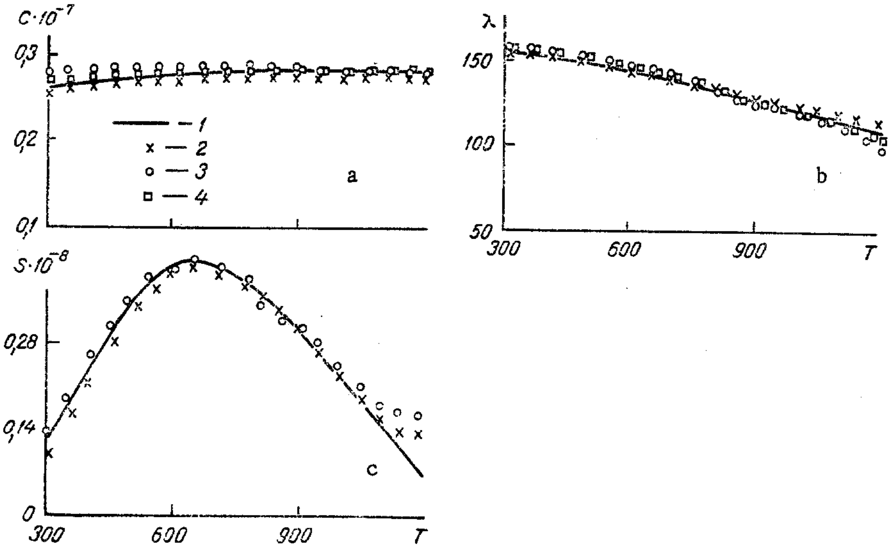

Fig. 1. Derivation of the dependences of the coefficients of the heat-conduction equation C, J/K•m³; λ, W/K; S, W/m³, on the temperature T, °K: 1) specified values; 2) derivation for accurate data; 3) for initial data specified with errors; 4) derivation of λ and C under the assumption that S is known.

The error in deriving the unknown coefficients, defined as

$$\delta_{u_j} = \int_{T_{\min}}^{T_{\max}} (u_j(T) - \tilde{u}_j(T))^2\, dT \Big/ \int_{T_{\min}}^{T_{\max}} (u_j(T))^2\, dT,\ j = \overline{1,\ 3}, \tag{26}$$

where u_j is the specified value and $\tilde{u}_j$ is the derived value, is (Fig. 1): $\delta_C = 0.03$, $\delta_\lambda = 0.04$; $\delta_S = 0.07$.

In processing the data of actual experiments, the initial data for IHP solution (thermosensor readings) are known with errors; therefore, the solution of the above-described IHP is modeled for initial data formed with the errors

$$f_i^*(\tau) = f_i(\tau)(1 + \beta\omega(\tau)),\ i = \overline{1,\ 3}, \tag{27}$$

where $f_i^*(\tau)$ are the thermosensor readings specified with errors; $f_i(\tau)$ are the accurate values of the thermosensor readings; $\omega(\tau)$ is a random quantity distributed according to a normal law N(0, 1); β is the relative maximum error of the calculations. In the calculations, β is taken to be 0.05.

The results of deriving the dependences C(T), λ(T), and S(T) using initial data specified with errors are shown in Fig. 1. The errors in deriving the true dependences are, respectively: $\delta_C = 0.06$; $\delta_\lambda = 0.06$; $\delta_S = 0.09$.

Note that the problem of determining the three coefficients of the heat-conduction equation is significantly less well founded than the problem of deriving the two coefficients C(T), λ(T). The results of determining C and λ under the assumption that S is known are shown in Fig. 1. The value of β is again taken to be 0.05; the error of the solution is $\delta_C = 0.045$, $\delta_\lambda = 0.048$.

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IDENTIFICATION OF BOUNDARY THERMAL PERTURBATIONS USING SPECTRAL FUNCTIONS

Yu. M. Matsevityi, A. P. Slesarenko, and O. S. Tsakanyan

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A method is proposed for solving the external inverse heat-conduction problem in a parametric formulation.

The heat-transfer boundary conditions at the surface may be determined from available information on the temperature inside an object, which forms a topic of the external inverse heat-conduction problem (IHP), by various methods [1-4], the application of which depends on the formulation of the IHP, the required accuracy of solution, and the presence of corresponding computational resources.

If the IHP is regarded as a control problem, in which the role of the control object is played by its model, the boundary conditions are taken as the input quantities, and the temperatures at the observation points as the output quantities, it is possible to speak of a correlation between the distributed input and output quantities, which may be expressed in the form of transfer functions or influence functions (the latter term, in our view, more closely corresponds to the physical meaning of this correlation).

In a particular case, determining the transfer functions of objects with distributed parameters (DP objects) consists in solving the heat-conduction equation for a single input perturbation at one of the boundary points, with zero perturbations at the other points of the surface [5]. The distributed transfer function from a single source at the given boundary point to a finite set of internal points of the given object is obtained here.

If, for each point boundary perturbation with amplitude f_i in the grid model of the object, the distributed function $W_i(x, y, z)$ is determined, where i = 1, 2, ..., N, the W_i functions may be used to write the relation between its temperature and all the input boundary perturbations for all its internal points, under the condition that the DP object is linear

$$T(x, y, z) = \sum_{i=1}^{N} f_i W_i(x, y, z).$$

In this case, solving the IHP reduces to determining the amplitudes f_i. Unique determination of the function f(x, y, z) entails having information on the temperatures at N internal points of the DP object and solving a system of N linear algebraic equations

$$T_j^* = \sum_{i=1}^{N} f_i W_{ij};\ j = 1, 2, \ldots, N,$$

where T_j^* are the temperatures at the observation points of the DP object.

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